Complex Analysis

First Test - November 3, 2014
Duration: 90 minutes

Notation: $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$ - the Riemann sphere; $\mathbb{H} = \{z \in \mathbb{C} : \Im z > 0\}$ - the upper half-plane; $\mathbb{D}(z_0, r) = \{z \in \mathbb{C} : |z - z_0| < r\}$; $\mathbb{D} = \mathbb{D}(0, 1)$ - the unit disc.

1. Let $E := \mathbb{R} \cup \{\infty\}$ be the real "circle" in $\mathbb{C}_\infty$, and $C := \partial\mathbb{D} = \{z \in \mathbb{C} : |z| = 1\}$ be the unit circle.
   (a) Consider the Möbius transformation $T(z) = \frac{z-i}{z+i}$. Show that $T(\mathbb{H}) = \mathbb{D}$ and that $T(E) = C$.
   (b) Prove that if F is a Möbius transformation such that $F(\mathbb{H}) = \mathbb{D}$, and $F(\infty) = 1$, then there is $\alpha \in \mathbb{H}$ such that $F(z) = \frac{z-\alpha}{z-\bar{\alpha}}$.

2. Let $f(z) = \frac{p(z)}{q(z)}$ be a rational function seen as a meromorphic function on $\mathbb{C}_\infty$.
   (a) If $\deg p \leq \deg q$, show that f has a removable singularity at ∞.
   (b) Suppose that $p(z)$ and $q(z)$ have no common roots and $c := \lim_{z\to\infty} f(z) \neq 0$. Show that
$$\frac{\prod_{z\in\mathbb{C}} p(z)^{\mathrm{ord}_z q}}{\prod_{z\in\mathbb{C}} q(z)^{\mathrm{ord}_z p}} = \pm c^{\deg p}.$$

3. Let Ω be a region in $\mathbb{C}$, $z_0 \in \Omega$ and $f \in H(\Omega)$.
   (a) Show that, for any $r > 0$ such that $\mathbb{D}(z_0, r) \subset \Omega$, Cauchy's integral formula:
$$f(z_0) = \frac{1}{2\pi i} \oint_{|z-z_0|=r} \frac{f(z)}{z - z_0} dz, \quad z \in \mathbb{D}(z_0, r)$$
   is equivalent to the mean value formula: $f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta}) d\theta$.
   (b) Suppose there is a non-empty disc $D := \mathbb{D}(z_0, r)$ such that $f(z) \in \overline{\mathbb{H}}$ for all $z \in D$. Show that, if $f(z_0) \in \mathbb{R}$, then f is constant in Ω.

4. (a) Let $\Omega = \mathbb{C} \setminus \{z_1, z_2\}$, $z_1 \neq z_2$, and let $f_i \in H(\Omega \cup \{z_i\})$ for $i = 1, 2$. Suppose that $\oint_{C_1} f_2(z)dz = \oint_{C_2} f_1(z)dz = w \in \mathbb{C}$, where C_1 and C_2 are two disjoint small curves inside Ω around z_1, z_2 respectively. Prove that the set
$$\left\{ \int_\gamma (\alpha_1 f_1(z) + \alpha_2 f_2(z))\, dz \mid \alpha_i \in \mathbb{Z},\ \gamma \text{ any closed curve in } \Omega \right\}$$
   equals the set $\{kw : k \in \mathbb{Z}\}$.
   (b) Let Ω be simply connected (every closed curve in Ω is homotopic to a point). If $g \in H(\Omega)$ and g never vanishes in Ω, show that there is $f \in H(\Omega)$ such that $g(z) = e^{f(z)}$ for all $z \in \Omega$. [Hint: consider the indefinite integral of $\frac{g'(w)}{g(w)}$].