Complex Analysis

Problem List 2 - Möbius Transformations

(Due Monday, 10/10)

1. If $T(z) = \frac{az+b}{cz+d}$ is a Möbius transformation which is not the identity map, and $ad - bc = 1$, show that $T \circ T(z) = z$, for every z, if and only if $a + d = 0$.

2. A meromorphic function $f \in \mathcal{M}(\mathbb{C}_\infty)$ is said to be holomorphic at ∞ if $\text{ord}_\infty f = \text{ord}_0 h \geq 0$ where $h(w) = f(\frac{1}{w})$; in this case, its derivative at ∞ is defined to be $f'(\infty) = h'(0)$. Show that a Möbius transformation $T(z)$ is holomorphic at ∞ if and only if $T(\infty) \neq \infty$, and in this case, show that $T'(\infty) \neq 0$.

3. Let T be a Möbius transformation with a unique fixed point $\alpha \in \mathbb{C}$. Show there exists $\beta \in \mathbb{C}^* \equiv \mathbb{C} \setminus \{0\}$, such that $\frac{1}{T(z)-\alpha} = \frac{1}{z-\alpha} + \beta$. Show that T is conjugated[1] to a translation of the form $S(z) = z + 1$.

4. Show that if T is a Möbius transformation with two distinct fixed points $\alpha, \beta \in \mathbb{C}$, then there exists $\lambda \in \mathbb{C}^*$, such that $\frac{T(z)-\alpha}{T(z)-\beta} = \lambda \frac{z-\alpha}{z-\beta}$. Show also that T is conjugated to a transformation of the form $S(z) = az$, $a \in \mathbb{C}^*$.

5. Consider the Möbius transformation T, such that $T(0) = 2, T(1) = 1, T(-1) = \frac{5}{3}$. How many fixed points does T have in $\mathbb{C}_\infty$? Determine $T(C)$, where C is the unit circle $C = \{z \in \mathbb{C} : |z| = 1\}$.

6. Show that $[z_1, z_2; z_3, z_4]$ is a real number if and only if the points z_1, z_2, z_3 and z_4 are in a circle of $\mathbb{C}_\infty$.

7. Show that the Möbius transformations that preserve the upper half-space $\mathbb{H}$ are the transformations of the form $T(z) = \frac{az+b}{cz+d}$ with $a, b, c, d \in \mathbb{R}$ and $ad - bc > 0$.

8. Let $T(z) = \dfrac{\alpha - z}{1 - \bar{\alpha} z}$ be a Möbius transformtion, with $\alpha \in \mathbb{C}$. What values α is not allowed to take? Show that T preserves the unit circle $C = \{z \in \mathbb{C} : |z| = 1\}$ and that, for $|\alpha| < 1$, $T(\mathbb{D}) = \mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

[1] Two Möbius transformations S and T are called conjugated if there exists another Möbius transformation F such that $S = F^{-1} \circ T \circ F$.