Complex Analysis

Problem List 3 - Local properties of holomorphic functions

(Due Wednesday, 19/10)

Throughout, Ω denotes an arbitrary region in $\mathbb{C}$.

1. Let f be holomorphic in Ω, and $z_0 \in \Omega$. Suppose $f(z) = \sum_{n\geq 0} a_n(z-z_0)^n$ is valid in some disc $D \subset \Omega$. Show that $f^{(n)}$ is a local isomorphism at z_0 if and only if $a_{n+1} \neq 0$.

2. Let $f \in H(\Omega)$ and $z_0 \in \Omega$. (a) Show that $\text{mult}_{z_0} f = \text{ord}_{z_0}(f(z) - f(z_0))$. (b) For a second function $g(z)$ holomorphic in a region containing $f(z_0)$ show that $\text{mult}_{z_0}(g \circ f) = \text{mult}_{z_0} f \cdot \text{mult}_{f(z_0)} g$.

3. Let $F(z, w)$ be a holomorphic function of two variables in Ω, that is $F(z, w)$ is holomorphic, as a function of $z \in \Omega$ when $w \in \Omega$ is fixed, and the same holds with z and w interchanged. Let $A \subset \Omega$ be a set with an accumulation point and suppose that $F(z, w) = 0$ for all $z, w \in A$. Show that $F(z, w) = 0$ for all $z, w \in \Omega$.

4. Let $f \in H(\Omega)$. Prove that if $\Re f$ or $\Im f$ (real and imaginary parts of f) have a local maximum at $z_0 \in \Omega$, then $f(z)$ is constant in Ω.

5. (Schwarz Lemma) Let f be a holomorphic function in the unit disc $\mathbb{D}$, with $f(0) = 0$ and $|f(z)| < 1$, for all $z \in \mathbb{D}$. Show that $g(z) = f(z)/z$ is a holomorphic function in $\mathbb{D}$ and show that $|f(z)| \leq |z|$ for all $z \in \mathbb{D}$.

6. Let f and g be two entire functions satisfying $|f(z)| \leq |g(z)|$ for all $z \in \mathbb{C}$. Show that there exists a constant $c \in \mathbb{C}$, with $|c| \leq 1$, such that $f(z) = cg(z)$. (Suggestion: use Riemann's removable singularities Theorem).

7. Let f be an entire function and let A, R and a be positive real numbers such that $|f(z)| \leq A|z|^a$, for all z with $|z| > R$. Prove that f is a polynomial of degree $n < a + 1$.