Complex Analysis

Problem List 5 - Localization of zeros and poles

(Due Wednesday, 9/11)

1. Let $f(z)$ be holomorphic in a region Ω satisfying $|f(z)-1|<1$, for all $z \in \Omega$. Show that $\displaystyle\int_\gamma \frac{f'(z)}{f(z)}dz = 0$, for any closed curve γ in Ω (not necessarily homologous to zero).

2. Let $f(z) = (z-z_0)^m h(z)$, where $m \in \mathbb{Z}$ and $h(z)$ is a holomorphic non-vanishing function in Ω. Let γ be a closed curve homologous to zero in Ω, with $z_0 \notin \{\gamma\}$. Show that
$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz = mI(\gamma, z_0).$$

3. Let γ be a simple closed curve in Ω, $f \in M(\Omega)$ ($f(z)$ is meromorphic in Ω) and let $g \in H(\Omega)$. If $\{z_1, ..., z_n\}$ is the set of zeros and poles of f, deduce the formula:
$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}g(z)dz = \sum_{k=1}^{n} g(z_k)\ \mathrm{ord}_{z_k} f.$$

4. Let γ be a simple closed curve in a region Ω, and $f \in H(\Omega)$, with n zeros (counting multiplicities) in the interior component of γ (and $f \neq 0$ on $\{\gamma\}$). Show that for any $g \in H(\Omega)$ there exists $\varepsilon > 0$ such that $f + \varepsilon g$ has also n zeros (counting multiplicities) in the interior component of γ.

5. Let Ω be a region in $\mathbb{C}$ and D an open disc such that $\overline{D} \subset \Omega$. Let $f \in H(\Omega)$ be not constant and such that $|f|$ is constant in ∂D. Show that f has at least one zero in D. (Suggestion: consider $g(z) = f(z) - f(z_0)$ with $z_0 \in D$).

6. Let $p_n(z) = \sum_{k=0}^{n} \frac{z^k}{k!}$ and $R > 0$, a fixed real number. Prove that there is an integer m such that p_n has no zeros in the disc of radius R for every $n \geq m$.

7. Consider the entire function $f(z) = z^{n+1} - e^{\frac{1}{2}z-1}$, for $n \geq 1$ integer. Show that f has $n+1$ zeros in the annulus $\left\{z \in \mathbb{C} : \frac{1}{2} < |z| < 1\right\}$ (observe that $e^{-\frac{5}{4}} > \frac{1}{4}$) and compute the index of the path around the origin given by $\gamma(t) = \frac{f(e^{it})}{e^{2it}}$, $t \in [0, 2\pi]$.