# Complex Analysis

## Problem List 6 - Conformal maps and Riemann's Theorem

## (Due Monday, 28/11)

Note: $\Re z$ and $\Im z$ denote, respectively, the real and imaginary parts of $z \in \mathbb{C}$.

1. Show that Riemann's theta function, defined by

$$\theta(z,\tau) = \sum_{n\in\mathbb{Z}} e^{\pi i\tau n^2 + 2\pi i n z}$$

is an entire function of the variable $z \in \mathbb{C}$, for $\tau \in \mathbb{H} = \{z \in \mathbb{C} : \Im z > 0\}$.

2. Let $\{f_n\}$ be a sequence of holomorphic functions on a region $\Omega$ which converges, uniformly on compact subsets $K \subset \Omega$ to a non-constant function $f$. Show that, if each $f_n$ is injective, then $f$ is also injective.

3. Let $f$ be holomorphic in the unit disc $\mathbb{D} = \mathbb{D}(0,1)$ satisfying $|f(z)| < 1$. Assuming that $f$ has two distinct fixed points $a$ and $b$ in $\mathbb{D}$, $f(a) = a$ and $f(b) = b$, show that $f(z) = z$ for all $z \in \mathbb{D}$.

4. Show that any injective holomorphic function $f : \mathbb{C}^* \to \mathbb{C}^*$ is of the form $f(z) = az$ or $f(z) = \frac{a}{z}$ for some $a \in \mathbb{C}^*$.

5. Determine explicitly a conformal map between the set $A = \{z \in \mathbb{C} : \Re z > 0, -\pi < \Im z < \pi\}$ and the unit disc $\mathbb{D}$.

6. Let $f$ be a conformal map from $A = \{z \in \mathbb{C} : \Re z > 0,\ \Im z > 0\}$ to the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, with the property $f(i+1) = 0$. Determine an explicit expression for $f$. Is there another map with these properties? Justify.

7. Let $\{f_n\}$ be a sequence of holomorphic functions in a region $\Omega$, uniformly bounded in compact subsets of $\Omega$. Show that, if $\lim_{n\to\infty} f_n(z)$ exists for all $z \in \Omega$, then $\{f_n\}$ converges uniformly in compact subsets of $\Omega$. (Suggestion: use the fact that $\{f_n\}$ is equicontinuous in compact subsets).

8. Let $B = \{z \in \mathbb{C} : |\Re z| < 1,\ |\Im z| < 1\}$. Show that there exists a conformal map between the set $\mathbb{C}\backslash\overline{\mathbb{D}}$ and the set $\mathbb{C}\backslash\overline{B}$.