Complex Analysis

Problem List 7 - Harmonic Functions and Factorization of Entire Functions

(Due Friday, 9/11)

1. Let $P_r(\theta)$ be the Poisson kernel, defined for $0 \leq r < 1$ and $\theta \in [0, 2\pi]$, by
$$P_r(\theta) = \frac{1}{2\pi} \frac{1 - r^2}{1 - 2r\cos\theta + r^2}.$$
Let $u \geq 0$ be a function continuous in $\overline{\mathbb{D}}$ and harmonic in $\mathbb{D} = \mathbb{D}(0, 1)$. Show the inequalities
$$\begin{aligned} \frac{1-r}{1+r} \leq &\ 2\pi P_r(\theta) &\leq \frac{1+r}{1-r} \\ \frac{1-r}{1+r}u(0) \leq &\ u(re^{i\theta}) &\leq \frac{1+r}{1-r}u(0). \end{aligned}$$

2. Consider a region $\Omega \subset \mathbb{C}$, a function u harmonic in Ω and $f \in H(\Omega)$. Assuming that $u = \Re f$ in a certain non-empty open disc $D \subset \Omega$, show that $u = \Re f$ throughout Ω.

3. (Harnak's Theorem) Let u_n be a sequence of harmonic functions in the unit disc $\mathbb{D}$ converging uniformly in compact subsets of $\mathbb{D}$. Show that the limit is a harmonic function on $\mathbb{D}$.

4. Show that u is harmonic in a region Ω if and only if it satisfies the "mean value property" for discs in Ω, which is: for every $z \in \Omega$ and disc $\mathbb{D}(z, r)$ whose closure is in Ω, one has:
$$u(z) = \frac{1}{\pi r^2} \int_{\mathbb{D}(z,r)} u\ dx\ dy.$$

5. Consider the function $f(z) = e^{2z} + e^{-2z} + 2$. Verify that f has simple zeros at the points $z_n = \frac{i\pi}{2}(2n + 1)$, and compute their orders. Determine the Hadamard factorization of f.

6. Let f be an entire function of order 1 whose zeros are simple and are located at the odd integers. Assuming that f is even, and that $\lim_{z \to 0} f(z) = 1$, determine the Hadamard factorization of f. Relating f with an elementary function show that
$$\prod_{n=2}^{\infty} \left(1 - \frac{4}{(2n-1)^2}\right) = \frac{1}{3}.$$

7. Let $f \in H(\mathbb{C})$ and n be a positive integer. Prove that there exists $g \in H(\mathbb{C})$ such that $g^n = f$ if and only if the orders of the zeros os f are divisible by n.

8. Let g be meromorphic in $\mathbb{C}$ with only simple poles and integer residues. Show that there exists a function f meromorphic in $\mathbb{C}$ such that $g = f'/f$.