Algebraic and Differential Topology

FCUL 2022/23

Problem Set 4 - Singular Homology

After exercise 7, you can assume Hurewicz theorem and homotopy invariance of H_k.

1. Show that $H_0(X)$ is isomorphic to the *abelian* group freely generated by the path components of X.

2. Let X be the union of path components $X_1, \cdots, X_m$. Prove that $H_k(X) = \oplus_{j=1}^{m} H_k(X_j)$ for all $k \in \mathbb{N}_0$.

3. Let $\gamma, \eta : I = [0,1] \to X$ be paths such that $\gamma(1) = \eta(0)$, and denote $\gamma^{-1}(t) := \gamma(1-t)$ (the inverse path). Prove that $[\gamma * \eta - \gamma - \eta]_\sim$ and $[\gamma * \gamma^{-1}]_\sim$ are zero in $H_1(X)$ (ie, the corresponding 1-chains are cycles which are homologous to zero).

4. Prove that $H_1(\mathbb{R}^n) = 0$ for all $n \in \mathbb{N}$. (Hint: use the previous exercise).

5. Let X be path connected and simply connected. Show that $H_1(X) = 0$.

6. (Reduced Homology) Let $\delta : \Delta_0(X) \to \mathbb{Z}$ be the degree map: $\delta(\sum a_j \sigma_j) = \sum a_j$. (a) Verify that $\delta \circ \partial_1 \equiv 0$ ($\partial_1 : \Delta_1(X) \to \Delta_0(X)$ is the usual boundary). (b) Define:

$$\widetilde{H}_0(X) := \frac{\ker \delta}{B_1(X)} = \frac{\ker \delta}{\operatorname{im} \partial_1}$$

   (called the 0th-reduced homology of X) and show that $\widetilde{H}_0(X) \cong \mathbb{Z}^{n-1}$ when $n \in \mathbb{N}$ is the number of path components of X.

7. Let Y be a path component of X and $f : Y \to X$ be the natural inclusion map. Prove that $f_*^{(k)} : H_k(Y) \to H_k(X)$ is an inclusion, for every $k \in \mathbb{N}_0$ and that there are homomorphisms $g_k : H_k(X) \to H_k(Y)$ such that $g_k \circ f_*^{(k)}$ is the identity map on $H_k(Y)$.

8. For a pointed space (X, x_0), let $\psi_X : \pi_1(X, x_0) \to H_1(X)$ be the Hurewicz map. Show that, for every morphism $f : (X, x_0) \to (Y, y_0)$ of path connected pointed spaces, the

diagram:

$$\begin{array}{ccc} \pi_1(X,x_0) & \stackrel{\psi_X}{\to} & H_1(X) \\ f_* \downarrow & & \downarrow f_* \\ \pi_1(Y,x_0) & \stackrel{\psi_Y}{\to} & H_1(Y) \end{array}$$

is commutative.

9. Let $X = \mathbb{R}^2 \setminus \{0\}$ and $\gamma : I \to X$ be given by $\gamma(t) = e^{2\pi i t}$. Prove that the 1-cycle γ is not homologous to zero on X, and conclude that $H_1(X) \neq 0$.

10. Compute H_1 of the following spaces: (a) a Möbius strip (b) An n dimensional torus $T^n := S^1 \times \cdots \times S^1$ (c) $\mathbb{R}^n \setminus \mathbb{R}^{n-2}$ $(n \geq 2)$ (d) $\mathbb{R}^3 \setminus \{(x,y,0) : x^2+y^2=1\}$

11. For the following maps f, determine f_* as homomorphisms of H_1 groups. (a) $f : S^1 \to S^1$, $z \mapsto z^n$, $n \in \mathbb{Z}$ (b) $f : S^1 \times S^1 \to S^1$, $(z_1, z_2) \mapsto z_1^a z_2^b$, $a, b \in \mathbb{Z}$.

12. Let $X = \mathbb{R}^2 \setminus \{p_1, p_2\}$ with $p_1 = (0,0)$ and $p_2 = (1,0)$, and $\Gamma : I \to X$ be the path whose image is in the picture below. Prove that $[\Gamma]_\sim = 0$ (Γ is homologous to zero as a 1-cycle) but that its class $[\Gamma] \in \pi_1(X, A)$ is non-trivial (Γ is not homotopic to a constant path), where A is the basepoint (initial and final point) (Hint for homology: decompose Γ into concatenated paths, and use Ex. 3 above; Hint for homotopy: what are the generators of $\pi_1(X,A)$?).