Complex Analysis

First Exam - January 11, 2016
Part I - Duration: 90 minutes

1. Let T be a Möbius transformation with a unique fixed point $\alpha \in \mathbb{C}$.
(a) Show that there exists a complex number $\beta \in \mathbb{C} \setminus \{0\}$ such that

$$\frac{1}{T(z) - \alpha} = \frac{1}{z - \alpha} + \beta,$$

for all $z \in \mathbb{C} \setminus \{\alpha\}$.
(b) Writing $T(z)$ as $\frac{az+b}{cz+d}$, prove that the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ has a single eigenvalue with one dimensional eigenspace.

2. Let $f(z)$ be a holomorphic function in $\mathbb{C}^*$ with a pole at $z = 0$.
(a) Show that $\lim_{|z|\to\infty} |f(z)| = \infty$ if and only if the *regular part* of the Laurent series development around $z = 0$, is a polynomial.
(b) Prove that $f(\mathbb{C}^*)$ is dense in $\mathbb{C}$ (without using Picard's theorems).

3. Let $p > 1$ be an integer, Ω a region in $\mathbb{C}$ and $f : \Omega \to \mathbb{C}$ an arbitrary function (not necessarily continuous). Suppose that $g(z) := f(z)^p$ and $h(z) := f(z)^{p+1}$ are both holomorphic in Ω.
(a) Prove that $f \in H(\Omega)$. [Hint: write $g(z) = (z - z_0)^k \tilde{g}(z)$, for $k = \text{ord}_{z_0} g$, and similarly for h].
(b) Give a counterexample to (a) assuming only $g \in H(\Omega)$ (but $h \notin H(\Omega)$).

4. Let $u : \Omega \to \mathbb{R}$ be infinitely differentiable, and suppose that $u(z) = u(r)$; that is, u only depends on $|z| = r$, where $z = re^{i\theta} \in \Omega$.
(a) Show that, for all $z \in \Omega$,

$$z\frac{\partial u}{\partial z} = \frac{1}{2} r u'(r).$$

[Hint: recall that $\frac{\partial}{\partial z} = \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y})$ and use the chain rule].
(b) Suppose that $u(z)$ is harmonic on $\mathbb{C}^*$ (and only depends on $|z|$). Prove that $u(z) = a + b \log |z|$, for some constants $a, b \in \mathbb{R}$.

Part II - Duration: 90 minutes
(One of the questions is optional)

1. Let $z_1, \cdots, z_n \in \mathbb{C}$ be distinct points and let $\Omega = \mathbb{C} \setminus \{z_1, \cdots, z_n\}$. Define C^Ω to be the set of all closed curves in Ω, and define the map $\psi : C^\Omega \to \mathbb{Z}^n$ by:

$$\psi(\gamma) = \frac{1}{2\pi i}\left(\oint_\gamma \frac{dz}{z - z_1}, \cdots, \oint_\gamma \frac{dz}{z - z_n}\right).$$

(a) Let $f \in H(\Omega)$ and $\gamma \in C^\Omega$ be fixed. Prove the statement: If $\psi(\gamma) = (0, \cdots, 0)$ then $\oint_\gamma f(z)dz = 0$. Is the converse true?
(b) Show that the image of ψ is the whole $\mathbb{Z}^n$.

2. Consider the polynomial $p(z) = 2z^5 + 6z - 1$.
(a) Show that $p(z)$ has a real root x with $0 < x < 1$. [Hint: check that $p'(x) > 0$ for $x \in \mathbb{R}$].
(b) Prove that $p(z)$ has, in the annulus $A = \{z \in \mathbb{C} :\ 1 < |z| < 2\}$, either four distinct simple zeros or two distinct double zeros.

3. Let $g(z)$ be an *even* entire function of order $\rho(g) \geq 1$, whose only zeros are simple and belong to $Z := \mathbb{Z} \setminus \{0\}$.
(a) Show that there exists an entire function $h(z)$ such that

$$g(z) = \frac{\sin(\pi z)}{z}\, e^{h(z^2)}.$$

(b) If $\rho(g) = 1$, and $g(0) = 2\pi$, determine $h(z)$.

4. Let $\wp(z)$ be the Weierstrass function relative to a lattice $\Lambda \subset \mathbb{C}$, and P be a fundamental polygon for Λ without zeros or poles of $\wp(z)$ in ∂P.
(a) Show that, for all $k \in \mathbb{N}$, the k-th derivative $\wp^{(k)}$ satisfies:

$$\sum_{z \in P} |\mathrm{ord}_z \wp^{(k)}| = 2(k+2).$$

(b) Show that there is a polynomial $q(z)$, of degree 2, such that $\wp''(z) = q(\wp(z))$, for all $z \in \mathbb{C} \setminus \Lambda$. [Hint: consider the Laurent series of $\wp''(z)$ around $z = 0$].

5. Let $C_r := \{z \in \mathbb{C} : |z| = r\}$ and let $F(z)$ be an entire function satisfying $F(C_1) \subset C_2$ and $F(C_2) \subset C_4$.
(a) Show that $F(C_{2^k}) \subset C_{2^{k+1}}$, for all $k \in \mathbb{Z}$.
(b) Assuming that the restriction of F to $\mathbb{D}$ is a conformal transformation between $\mathbb{D}$ and $\mathbb{D}_2 = \{z \in \mathbb{C} : |z| < 2\}$, show that there exists $c \in \mathbb{C}$ with $|c| = 2$, such that $F(z) = cz$.