Complex Analysis

First Test - Oct. 31, 2015

1. Let $T(z) = \frac{\alpha - z}{1-\bar{\alpha}z}$ be a Möbius transformation, with $\alpha \in \mathbb{C}$.
(a) What are the values α is not allowed to take?
(b) Show that T maps the unit circle $C = \{z \in \mathbb{C} : \ |z| = 1\}$ to itself.
(c) In case $|\alpha| < 1$, show that $T(\mathbb{D}) = \mathbb{D} := \{z \in \mathbb{C} : \ |z| < 1\}$.
Possible solution:
(a) Since $T(z) = \frac{\alpha - z}{1-\bar{\alpha}z}$ is a Möbius tranformation, we have $-\alpha\bar{\alpha} + 1 \neq 0$ or equivalently $|\alpha|^2 \neq 1$. So, α is not allowed to have unit norm.
(b) Suppose that $z \in C$, that is $|z| = 1$. Then:
$$|T(z)| = \frac{|\alpha - z|}{|1 - \bar{\alpha}z|} = \frac{|\bar{z}|\,|\alpha - z|}{|1 - \bar{\alpha}z|} = \frac{|\alpha\bar{z} - 1|}{|\bar{\alpha}z - 1|} = \frac{|\overline{\bar{\alpha}z - 1}|}{|\bar{\alpha}z - 1|} = 1.$$
This shows that $T(C) \subset C$. As T is a bijective map from $\mathbb{C}_\infty$ to itself, and the image of a given circle in $\mathbb{C}_\infty$ is a set of the same kind, we conclude that $T(C) = C$.
(c) If $|\alpha| < 1$, then $|T(0)| = |\alpha| < 1$. We have $\mathbb{C}_\infty = \mathbb{D} \sqcup C \sqcup \mathbb{D}_\infty$ (disjoint unions) where $\mathbb{D}_\infty := \{z \in \mathbb{C} : |z| > 1\} \cup \{\infty\}$. Because T is bijective and continuous, and $T(C) = C$, by part (b), we have either $T(\mathbb{D}) = \mathbb{D}$ or $T(\mathbb{D}) = \mathbb{D}_\infty$. This follows from the fact that T sends connected sets to connected sets, noting that $\mathbb{D} \sqcup \mathbb{D}_\infty$ is the decomposition of $\mathbb{C}_\infty \setminus C$ into connected open sets (the north and south hemispheres of the Riemann sphere). Since $0 \in \mathbb{D}$ and $T(0) = \alpha \in \mathbb{D}$, we conclude that $T(\mathbb{D}) = \mathbb{D}$.

2. Let $f(z)$ be a holomorphic function in $\mathbb{C} \setminus \{z_0\}$ with a pole at $z_0 \in \mathbb{C}$, and let $g(z) := f(\frac{1}{z} + z_0)$.
(a) Prove that $g(z)$ is meromorphic in $\mathbb{C}$ if and only if $f(z)$ is a rational function.
(b) In the case that $\lim_{|z|\to\infty} |f(z)| = 0$, show that $g(z)$ is a polynomial, and zero is one of its roots.
Possible solution:
(a) If z_0 is a pole of order $k > 0$ of the function $f(z)$, then:
$$f(z) = \frac{b_k}{(z - z_0)^k} + \cdots + \frac{b_1}{z - z_0} + a_0 + a_1(z - z_0) + \cdots + a_n(z - z_0)^n + \cdots ,$$

is the correspondent Laurent development, valid in $\mathbb{C} \setminus \{z_0\}$. So, we have:

$$g(z) = f(\frac{1}{z} + z_0) = b_k z^k + \cdots + b_1 z + a_0 + \frac{a_1}{z} + \cdots + \frac{a_n}{z^n} + \cdots , \tag{1}$$

valid in $\mathbb{C} \setminus \{0\}$. The origin is an isolated singularity of $g(z)$, and it is a pole if and only if the set $\{a_1, a_2, \cdots\}$ has only a finite number of non-zero terms. Equivalently, g is meromorphic in $\mathbb{C}$ iff there is a natural number m such that $a_j = 0$ for all $j > m$. In this case, the series above is finite, so:

$$f(z) = \frac{b_k + \cdots + b_1(z - z_0)^{k-1} + a_0(z - z_0)^m + \cdots + a_m(z - z_0)^{k+m}}{(z - z_0)^k},$$

which is clearly a rational function.

(b) The equation $g(z) = f(\frac{1}{z} + z_0)$ is equivalent to $f(z) = g(\frac{1}{z-z_0}) = g(w)$, for $z \neq z_0$, where we use $w := \frac{1}{z-z_0}$. Moreover, using equation (1):

$$\lim_{|z|\to\infty} |f(z)| = \lim_{|\frac{1}{z-z_0}|\to 0} |g(\tfrac{1}{z-z_0}))| = \lim_{|w|\to 0} |g(w)| =$$
$$= \lim_{|w|\to 0} \left| b_k w^k + \cdots + b_1 w + a_0 + \frac{a_1}{w} + \cdots + \frac{a_n}{w^n} + \cdots \right|$$

By the classification of singularities, this limit exists (and is finite) if and only if $w = 0$ is a removable singularity of g, so that $a_1 = a_2 = \cdots = 0$. Moreover, this limit is zero iff also $a_0 = 0$. This means that $g(w) = b_k w^k + \cdots + b_1 w$ is a polynomial without constant term. Equivalently, 0 is a root of g.

3. Let $f(z)$ be an entire function ($f \in H(\mathbb{C})$). Suppose that there is $w_0 \in \mathbb{C}$ and a neighborhood U of w_0 such that $f(\mathbb{C}) \cap U$ is empty (in other words, the image of f does not intersect U). Show that $f(z)$ is constant. [Hint: consider the function $h(z) := \frac{1}{f(z)-w_0}$].

Possible solution:

Let U be a neigborhood of z_0 such that $f(\mathbb{C}) \cap U$ is empty. Let $\mathbb{D}(w_0, r)$ be a disc completely contained in U, with $r > 0$. Let

$$h(z) = \frac{1}{f(z) - w_0}.$$

Since $f(z) = w_0$ is an equation without solution $z \in \mathbb{C}$, $h(z)$ is also an entire function (it is a composition of an entire function with a rational function whose denominator does not vanish). Moreover, since $f(\mathbb{C}) \cap \mathbb{D}(w_0, r) = \emptyset$ we have the inequality

$$|f(z) - w_0| > r, \quad \forall z \in \mathbb{C}.$$

This implies the bound:

$$|h(z)| = \frac{1}{|f(z) - w_0|} < \frac{1}{r}.$$

Finally, by Liouville's theorem, we conclude that h is constant, which means that f is also constant.

4. Let $u(z)$ be a harmonic function in the plane $\mathbb{C}$.
(a) Show that, for any radius $R > 0$ and $z \in \mathbb{C}$ with $|z| = r < R$, the following inequality holds:

$$\frac{R-r}{R+r}u(0) \leq u(z) \leq \frac{R+r}{R-r}u(0).$$

[Hint: Estimate the Poisson kernel for $\mathbb{D}(0,R)$: $P_{R,r}(\alpha) = \frac{1}{2\pi}\Re\left(\frac{Re^{i\alpha}+r}{Re^{i\alpha}-r}\right)$ and use Poisson's formula in this case: $u(re^{i\theta}) = \int_{-\pi}^{\pi} P_{R,r}(\theta-\tau)\, u(Re^{i\tau})\, d\tau$].
(b) If $u(z)$ is bounded ($\exists M > 0$ such that $|u(z)| \leq M$ for all $z \in \mathbb{C}$), show that $u(z)$ is constant.

Possible solution:

We can write:

$$2\pi P_{R,r}(\alpha) = \Re\left(\frac{Re^{i\alpha}+r}{Re^{i\alpha}-r}\right) = \Re\left(\frac{(Re^{i\alpha}+r)(Re^{-i\alpha}-r)}{(R\cos\alpha - r)^2 + R^2\sin^2\alpha}\right) = \frac{R^2-r^2}{R^2 - 2Rr\cos\alpha + r^2}.$$

Using that $R < r$, we obtain $(R-r)^2 \leq R^2 - 2Rr\cos\alpha + r^2 \leq (R+r)^2$, for any real angle$\alpha$. This gives the inequalities:

$$\frac{R^2-r^2}{(R+r)^2} = \frac{R-r}{R+r} \leq 2\pi P_{R,r}(\alpha) \leq \frac{R+r}{R-r} = \frac{R^2-r^2}{(R-r)^2}.$$

Using the integral formula indicated, and the mean value property $2\pi u(0) = \int_{-\pi}^{\pi} u(Re^{i\tau})\, d\tau$, we get:

$$2\pi u(0)\frac{R-r}{R+r} = \int_{-\pi}^{\pi} \frac{R-r}{R+r}u(Re^{i\tau})\, d\tau \leq \int_{-\pi}^{\pi} 2\pi P_{R,r}(\theta-\tau)\, u(Re^{i\tau})\, d\tau = u(z).$$

The other inequality being analogous.

(b) By adding M to u, we keep harmonicity, so we can show that $0 \leq u(z) \leq 2M$, for all $z \in \mathbb{C}$ implies that u is constant. Fiz a $z = re^{i\theta} \in \mathbb{C}$. By part (a) we have

$$0 \leq u(z) \leq \frac{R+r}{R-r}u(0), \quad \text{for all } R > 0.$$

By letting $R \to \infty$ we see that $u(z) \leq u(0)$ for all $z \in \mathbb{C}$. So, u reaches a maximum at $z = 0$, which by the maximum principle, proves that u is constant.