Complex Analysis

First Test - November 9, 2011

Duration: 90 minutes

(One of the questions is optional)

1. Let $F(z)$ be a Möbius transformation that preserves the real "circle" $E := \mathbb{R} \cup \{\infty\}$.
   (a) Show that we can write $F(z) = \frac{az+b}{cz+d}$, with $a, b, c, d \in \mathbb{R}$.
   (b) Assume $a, b, c, d \in \mathbb{R}$, and $ad - bc = 1$. Define $\tau := a + d$. Show that $F(z)$ has 2 real fixed points if $|\tau| > 2$, and 2 conjugate non-real fixed points if $|\tau| < 2$. In the case $|\tau| > 2$, prove that there is $\rho > 0$ ($\rho \neq 1$) and a Möbius transformation $S(z)$ such that $S(F(S^{-1}(z))) = \rho z$.

2. Let $f(z)$ be a meromorphic function in $\mathbb{C}$.
   (a) If all singularities of f are removable, and $f(\mathbb{C})$ avoids a disc $D = \mathbb{D}(w_0, r)$, $r > 0$ (so that $f(z) \notin D$, for all $z \in \mathbb{C}$), show that $f(z)$ is constant. (Suggestion: consider the function $h(z) = \frac{1}{f(z) - w_0}$).
   (b) Suppose that there exist $R, M > 0$ such that $|f(z)| \leq M$ for all z such that $|z| > R$. Show that $f(z)$ is a rational function.

3. Let $f, g \in H(\Omega)$ and let $D := \mathbb{D}(z_0, R)$ be a disc whose closure is in Ω. Assume that f and g have no zeros in D, and that $|f(z)| = |g(z)|$ for all z in the boundary circle $C := \{z \in \mathbb{C} : \ |z - z_0| = R\}$. Prove that there is a constant λ of modulus 1, such that $f(z) = \lambda g(z)$, for all $z \in \Omega$.

4. Let $\Omega = \mathbb{C} \setminus \{z_1, ..., z_n\}$. Define a relation in $H(\Omega)$ as follows. For $f_1, f_2 \in H(\Omega)$, we put $f_1 \sim f_2$ if $f_1(z) - f_2(z)$ admits an antiderivative in Ω.
   (a) Show that $\sim$ is an equivalence relation, and that $H(\Omega)/\!\sim$ is a vector space.
   (b) Let $\gamma_1, ..., \gamma_n$ be the boundaries of small disjoint discs $D_1, ..., D_n$ centered at $z_1, ..., z_n$ (respectively, oriented counterclockwise), and consider the map $\phi : H(\Omega)/\!\sim \rightarrow \mathbb{C}^n$ defined by $[f] \mapsto (\int_{\gamma_1} f(z)dz, \cdots, \int_{\gamma_n} f(z)dz)$. Show that ϕ is a well defined linear map, and that it is an isomorphism of vector spaces.

5. Let $n > 0$ be an integer.
   (a) Compute the integral $\oint_{|z|=1} \frac{e^z + 3nz^{n-1}}{e^z + 3z^n} \, dz$, where the circle goes around $z = 0$ one time in the counterclockwise direction.
   (b) Use the estimate $\frac{3}{4^n} < e^{-\frac{1}{4}}$ to show that the function $g(z) = e^z + 3z^n$ has no zeros in the disc $\mathbb{D}(0, \frac{1}{4})$.